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An Approximation Theorem

LIPMAN BERS

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AN APPROXIMATION THEOREM*

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The simple approximation theorem stated below, an incidental by product of an investigation with a different aim, seems not to be recorded in the literature. The proof uses a device due to Ahlfors.

Definition. Let D be a domain in the complex plane, $\dot{D}$ its boundary and $\Lambda \subset \dot{D}$ a closed set. We call Λ *ample* if (i) it contains every point of $\dot{D}$ which is not a boundary point of the complement G of $D \cup \dot{D}$, and (ii) in every component of G, the part of Λ contained in its boundary has positive harmonic measure.

Examples. Let D be the complement of a nowhere dense set Λ; then Λ is ample. Let D be bounded by k closed Jordan curves C_j and let λ_j be a subarc of Λ_j; then $\Lambda = \lambda_1 \cup \ldots \cup \lambda_k$ is ample. If C_j is rectifiable, it suffices to assume that $\lambda_j \subset C_j$ has positive linear measure.

Theorem. *Let Λ be a set on the boundary of a plane domain D and assume that the closure of Λ is ample. Let f(z) be analytic in D and such that*

$$\iint_D |f(z)| \, dx \, dy < +\infty. \tag{1}$$

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<u>Then there exists a sequence of rational functions</u> $r_j(z)$, <u>with simple poles in</u> Λ <u>and no other singularities, such that</u>

$$(2) \qquad \lim_{j\to\infty} \iint_D |f(z) - r_j(z)| \, dx\, dy = 0.$$

<u>Proof.</u> We assume Λ to be infinite; otherwise the statement is trivial. Let α denote the set of rational functions with simple poles in Λ, which are absolutely integrable over D. Analytic functions satisfying (1) form a Banach space. Let ℓ be a continuous linear functional on this space. It suffices to show that if $\ell(\phi) = 0$ for all ϕ in α, then $\ell \equiv 0$.

Every ℓ is of the form

$$(3) \qquad \ell(f) = \iint_D f(z)\, \mu(z)\, dx\, dy$$

where μ is a bounded measurable function. Let a_1 and a_2 be two points in Λ and set

$$(4) \qquad h(z) = -\frac{(z - a_1)(z - a_2)}{\pi} \iint_D \frac{\mu(\zeta)\, d\xi\, d\eta}{(\zeta - z)(\zeta - a_1)(\zeta - a_2)}$$

Then $h(z)$ is continuous everywhere, $h(a_1) = h(a_2) = 0$, h has generalized derivatives which are locally square integrable,

$$(5) \qquad \partial h / \partial \bar{z} = \mu \text{ in D}$$

and $h(z)$ is analytic in the complement G of the closure of D. Also,

(6) $$h(z) = O(|z| \log |z|), \quad z \to \infty,$$

and, for every $R > 0$,

(7) $$|h(z') - h(z'')| \leqslant C(R)\, |z' - z''| \, |\log |z' - z''|| \quad \text{for } |z'|, |z''| \leqslant R.$$

All this is verified by standard arguments.

Assume that $\ell(\phi) = 0$ for all ϕ in α. For every $a \in \Lambda$, $a \neq a_1, a_2$, the function $\phi(\zeta) = (\zeta - a)^{-1}(\zeta - a_1)^{-1}(\zeta - a_2)^{-1}$ belongs to α. For this ϕ, $-\pi h(a) = (a - a_1)(a - a_2)\ell(\phi)$. Thus $h = 0$ on the closure of Λ. Using condition (ii) of ampleness we conclude that $h \equiv 0$ in G, and hence

(3) $$h = 0 \text{ on } \dot{D}$$

Let $\delta(z)$ denote the distance from z to $\dot{D}$; by (6) and (3)

(9) $$|h(z)| \leqslant C(R)\, \delta(z) \, |\log \delta(z)| \text{ for } |z| \leqslant R.$$

Now let $j(t)$, $-\infty$ be an infinitely differentiable function such that $0 < j(t) < 1$, $j(t) = 0$ for $t \leqslant 1$, $j(t) = 1$ for $t > 1$ and set, for $n = 1, 2, \ldots,$ and for z in D,

$$\omega_n(z) = j\left(-n / \log\log \frac{1}{\delta(z)}\right)$$

(this device is due to Ahlfors). Since $\delta(z)$ is Lipschitz continuous with constant 1, and $j'(t) = 0$ outside the interval $1 < t < 2$, one verifies that

(10) $$\left| \frac{\partial \omega_n(z)}{\partial z} \right| \leqslant \frac{c}{n} \; \frac{}{\delta(z) \, |\log \delta(z)|}.$$

For every $R > 0$, let $D(R)$ and $\Gamma(R)$ denote the intersection of D with the disc $|z| < R$ and the circle $|z| = R$, respectively. By (5) and Stokes' theorem

$$\iint\limits_{D(R)} \omega_n(z) f(z) \mu(z)\, dx\, dy = \iint\limits_{D(R)} \omega_n(z) \frac{\partial}{\partial \bar z} (h(z) f(z))\, dx\, dy$$

$$= -\frac{i}{2} \int\limits_{\Gamma(R)} \omega_n(z) h(z) f(z) dz - \iint\limits_{D(R)} f(z) h(z) \frac{\partial \omega_n(z)}{\partial \bar z}\, dx\, dy$$

for every $f(z)$ analytic in D, since $\omega_n \equiv 0$ near $\dot D$. Assume now that (1) holds. By (9) and (10) the last integral goes to 0 for $n \to \infty$, and, since $\omega_n \to 1$, we conclude that

$$\left| \iint\limits_{D(R)} f(z) \mu(z)\, dx\, dy \right| \leq \left| \frac{1}{2} \int\limits_{\Gamma(R)} f(z) h(z)\, dz \right|.$$

Here the right hand side vanishes for large R if D is bounded, but is in all cases less than

$$\text{(11)} \qquad \text{const. } R \log R \int\limits_{\Gamma(R)} |f(z)|\,|dz| \qquad (R > 1)$$

in view of (6). Since

$$\int_0^{+\infty} \left\{ \int\limits_{\Gamma(R)} |f(z)|\,|dz| \right\} dR < +\infty$$

by (1), the quantity (11) can not remain above a positive number as $R \to \infty$. We conclude from (3) that $\ell(f) = 0$, q.e.d.

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